UNIVERSITÀ DEGLI STUDI DI TORINO

I@UNITO – Visiting Scientists

Scientific area	Scientific responsible	Host Department	Type of activity	Start of mobility	Language
Area 1 (Analysis MAT/05)	S. Coriasco	Mathematics “G. Peano”	Basic research	February 2017 (tentatively)	English

Type of fellowship	Junior (less than 40 years old) 3 months fellowship
Title of the research project	Fourier integral operators, symplectic geometry and analysis on noncompact manifolds
Description of the research project	In this research project, we aim at establishing a generalization of Fourier integral operators methods to an asymptotically Euclidean setting, continuing and extending the analysis started in [10, 11]. We first recall some basic elements of the theory of Fourier integral operators, see [12, 15, 25], and then indicate the new aspects of the theory we will develope. In short, local Fourier integral operators are defined as those linear operators A, acting from the smooth compactly supported functions on an open subset Y to the smooth functions on an open subset X, whose kernels are oscillatory integrals, associated with a phase function φ and a symbol (amplitude) of order μ. Usually, the phase function φ satisfies a non-degeneracy conditions. In the latter case, A is a bounded operator between the mentioned smooth functions spaces, extendable to a continuous operator $A\colon \mathcal{E}'(Y) \to \mathcal{D}'(X)$. In particular, when the dimensions of X, Y, and the “frequency space” $\mathbf{R}^N$ coincide, and $\varphi(x, y, \theta) = (x - y) \cdot \theta$, A is a pseudodifferential operator of order μ. This local definition can be extended to manifolds, by means of the concept of Lagrangian distribution, see, e.g., [15, 23]. Namely, given two smooth, closed manifolds X and Y and a smooth closed conic Lagrangian submanifold $\Lambda \subset T^*(X \times Y) \setminus 0$, an integral operator A with kernel $K_A \in I^m(X \times Y, \Lambda)$, the space of Lagrangian distributions of order m on $X \times Y$ associated with Λ, is a Fourier Integral Operator of order m if $\Lambda \subset \{(x,y,\xi,\eta) \in T^*(X \times Y) \setminus 0\colon \xi \neq 0, \eta \neq 0\}$. In such case, one simply writes $A \in I^m(X \times Y, \Lambda)$. It turns out that, in local coordinates on X and Y, the kernels of the globally defined Fourier integral operators are given, modulo smooth remainders, by the oscillatory integrals appearing in the local definition above. Moreover, the non- degenerate phase function φ locally parametrizes Λ, in the sense that, setting $$\Sigma_\varphi = \{(x,y,\theta)\colon \varphi'_\theta(x,y,\theta) = 0\},$$ the map $(x,y,\theta) \mapsto (x,y,\varphi'_{x,y}(x,y,\theta))$ is a local homogeneous diffeomorphism of Σ_φ onto Λ. The principal symbol of A can also be invariantly defined. An important subclass of Fourier integral operators consists of all $A \in I^m(X \times Y,\Lambda)$ such that Λ is the graph of a homogeneous symplectomorphism $\chi : T^*X \setminus 0 \to T^*Y \setminus 0$, that is, the canonical transformation χ commutes with multiplication by positive constants in the fiber. In this case, of course, dim X = dim Y. A calculus for these global operators can be established, see [15, 23]. In particular, a composition theorem can be proved, properties of the adjoint operators and rules for the computation of the principal symbols of $A_1 \circ A_2$ and A^* can be given as well. Other important aspects of the theory concern the propagation of wave front sets and the boundedness on different functional spaces. These can be applied to the study of the regularity of solutions of Cauchy problems associated with hyperbolic equations, see, for instance, the celebrated theorem

by A. Seeger, C.D. Sogge and E.M. Stein [22].

The theory of Fourier Integral Operators is still an interesting field of active research, with developments in various directions, see, e.g., [2, 3, 7, 9, 16, 20, 24], and the references quoted therein. In the present project we deal with the class of so-called SG Fourier integral operators, initially considered in [5, 6], see also [1, 7, 9]. Such classes of Fourier operators naturally appear in the solution of Cauchy problems for hyperbolic differential operators on $\mathbf{R}^n$ with polynomially bounded coefficients. The symbol estimates involve the global behavior also with respect to the x variable, namely, for any $\alpha,\beta \in Z^n_+$ there exists $C_{\alpha\beta} > 0$ such that, for all $x,\xi \in \mathbf{R}^n$,

$$|D_x^{\alpha} D_\xi^{\beta} a(x,\xi)| \le C_{\alpha\beta}(1+|x|)^{m-|\alpha|} (1+|\xi|)^{\mu-|\beta|},$$

for some constants $m,\mu \in \mathbf{R}$. The aim of this project is to look for further extensions of the results described in the papers mentioned above.

In particular, a satisfying definition of SG Fourier integral operator in terms of kernels as in the definitions recalled above is still missing. Such a definition would allow the extension of the SG Fourier integral operators theory to a natural class of manifolds where the SG calculus can be transferred, namely, the manifolds with ends, see [4, 8, 17]. A tool which looks very promising to such aim is the calculus of the subclass of the classical SG symbols. Initial results have been obtained in [10, 11] (see also [1, 21]). These include the analysis of a temperate version of the oscillatory integral kernels and of their (global) wave-front sets, see [10], as well as the investigation of the natural “symplectic structure at infinity” arising in this context, and the parametrization of the corresponding analogue of the Lagrangian submanifolds, given in [11]. In the latter, a main ingredient of the analysis is the differential calculus on manifolds with corners (cfr. [18]).

More precisely, the approach pursued in [11] is a further generalization of the classical theory in terms of the SG calculus on $\mathbf{R}^n$, focusing on the properties of the involved phase functions and of the corresponding generalized Lagrangian submanifolds. The advantage is that the results can be formulated in terms similar to the classical ones, described above, while still allowing a broad class of phase functions and including “singularities at infinity”. An example of a distribution that may be treated from this point of view is the so-called two-point function, arising in the study of the Klein-Gordon equation. We note that the approach of [13, 14, 17], which is formulated in the language of sc-geometry on asymptotically flat, or scattering, manifolds, while being related to the analysis in [11], is different from it. A major distinction is that the typical phase functions in [11] give rise to Lagrangian type singularities in all three components of the compactified cotangent bundle and the associated distributions are not smooth functions like the Legendrian distributions in [19]. In fact, the above mentioned two-point function is not a smooth function, thus not a Legendrian distribution in the sense of [19], but admits Lagrangian type singularities in the interior as well as Legendrian type singularities at infinity, see [11] for details.

In [10] a theory of tempered oscillatory integrals has been established, which may be viewed as the local version of distributions arising from the geometric structures “at infinity” mentioned in the previous paragraph. The involved objects extend the theory of classical oscillatory integrals, in the sense that they are tempered, and that their global singularities may be understood in terms of the global set of stationary points of their phase functions. The phase functions are assumed to be (inhomogeneous) SG symbols, whose derivatives satisfy an ellipticity condition.

In [11] the theory is complemented with the geometric picture, under the (natural) additional assumption that the phase function φ is SG classical, that is a SG symbol of order (1, 1) which admits polyhomogeneous expansions. We note that even in this case the distributions under consideration differ from Legendrian distribution. In fact, by [19, Proposition 10], the singularities of the Fourier transforms of Legendrian distributions on Euclidean spaces are contained in compact sets, a feature that is not true for the class of distributions studied in [10]. It is also discussed how the global set of stationary points of a non-degenerate SG classical phase functions form generalized Lagrangian submanifolds, which are submanifolds of a compactification of $T^*\mathbf{R}^n$, a manifold with corners, which turns out to be the natural environment within which to perform this analysis. In particular, it is shown that the generalized Lagrangian submanifolds mentioned above can always be parametrized by SG classical phase functions, and examine when two such parametrizations may be regarded as equivalent.

Further developments will be pursued within this project, by addressing the actual calculus of SG Lagrangian distributions and corresponding Fourier integral operators, with emphasis to the principal symbol maps and applications to hyperbolic differential equations.

A useful tool to investigate singularities in Euclidean spaces is the FBI-transform. This transform may be used to characterize microlocal singularities by means of time-frequency analysis, and there are adapted version of it suitable for global calculi. In particular, it can be used to investigate singularities that arise in both the SG- as well as the Shubin calculus. A thorough understanding of conormal and Lagrangian singularities by means of such integral transforms, and hence applications to the theory of Fourier integral operators, remains to be achieved. Such topics will also be investigated within this project.

References

1. G.D. Andrews. A closed class of SG Fourier integral operators with appli- cations. Ph.D. Thesis, Imperial College, London (2004)
2. U. Battisti, S. Coriasco, E. Schrohe. Fourier integral operators and the index of symplectomorphisms on manifolds with boundary. J. Funct. Anal. (2015) DOI 10.1016/j.jfa.2015.06.001.
3. E. Cordero, F. Nicola and L. Rodino. Boundedness of Fourier integral Operators on FL^p spaces. Trans. Amer. Math. Soc., 361 (2009), 6049– 6071.
4. H. O. Cordes, The Technique of Pseudodifferential Operators. Cambridge Univ. Press (1995).
5. S. Coriasco. Fourier Integral Operators in SG Classes II: Application to SG Hyperbolic Cauchy Problems. Ann. Univ. Ferrara, Sez. VII - Sc. Mat. 44 (1998), 81-122 (1999).
6. S. Coriasco. Fourier Integral Operators in SG Classes I: Composition Theorems and Action on SG Sobolev Spaces. Rend. Sem. Mat. Univ. Politec. Torino. 57 (1999), 4, 249-302 (2002).
7. S. Coriasco, K. Johansson, J. Toft. Global wave-front properties for Fourier integral operators and hyperbolic problems. J. Fourier Anal. Appl. (2015) DOI 10.1007/s00041-015-9422-1.
8. S. Coriasco, L. Maniccia, On the Spectral Asymptotics of Operators on Manifolds with Ends. Abstr. Appl. Anal. vol. 2013, Article ID 909782, DOI 10.1155/2013/909782 (2013).
9. S. Coriasco and M. Ruzhansky. Global L^p continuity of Fourier integral operators. Trans. Amer. Math. Soc. 366, 5 (2014), 2575-2596.

	10. S. Coriasco, R. Schulz. The global wave front set of tempered oscillatory integrals with inhomogeneous phase functions. J. Fourier Anal. Appl. 19, 5 (2013), 1093-1121. 11. S. Coriasco, R. Schulz. SG-Lagrangian submanifolds and their parametrization. Preprint, arXiv:1406.1888 (2014). Submitted. 12. J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. Acta Math., 128(3-4):183–269, 1972. 13. A. Hassel and A. Vasy, The spectral projections and the resolvent for scat- tering metrics, J. Anal. Math., 79 1 (1999), 241-298. 14. A. Hassel and J. Wunsch, The semiclassical resolvent and the propagator for non-trapping scattering metrics, Adv. Math. 217 (2008), no. 2, 586–682. 15. L. Hörmander. Fourier integral operators. I. Acta Math., 127(1-2):79–183, 1971. 16. E. Leichtnam, R. Nest, and B. Tsygan. Local formula for the index of a Fourier integral operator. J. Differential Geom., 59(2):269–300, 2001. 17. R. Melrose, Geometric scattering theory. Stanford Lectures. Cambridge University Press, Cambridge (1995). 18. J. Margalef-Roig and E. Outerelo Dominguez, Differential topology, Elsevier, 1992. 19. R. Melrose and M. Zworski, Scattering metrics and geodesic flow at infinity, Inventiones Mathematicae 124 (1-3) (1996), 389-436. 20. V. E. Nazaikinskii, B.-W. Schulze, and B. Y. Sternin. The index of Fourier integral operators on manifolds with conical singularities. Izv. Ross. Akad. Nauk Ser. Mat., 65(2):127–154, 2001. 21. R. Schulz, Microlocal Analysis of Tempered Distributions, Diss. Niedersächsische Staats-und Universitätsbibliothek Göttingen, 2014. 22. A. Seeger, C.D. Sogge and E.M. Stein. Regularity properties of Fourier integral operators. Ann. of Math. 134 (1991), 231–251. 23. C.D. Sogge. Fourier integrals in classical analysis. Cambridge University Press, 1993. 24. A. Weinstein. Fourier integral operators, quantization, and the spectra of Riemannian manifolds. In Géométrie symplectique et physique mathématique (Colloq. Internat. CNRS, No. 237, Aix-en-Provence, 1974), pages 289–298. Editions Centre Nat. Recherche Sci., Paris, 1975. With questions by W. Klingenberg and K. Bleuler and replies by the author. 25. K. Yagdjian. The Cauchy problem for hyperbolic operators, volume 12 of Mathematical Topics. Akademie Verlag, Berlin, 1997. Multiple characteristics, Micro-local approach.
Profile Description	It is a requested prerequisite that the visiting scientist has a strong background in microlocal analysis techniques, in particular on global calculi, Fourier integral operators and PDE analysis on noncompact settings ($\mathbf{R}^n$ and/or noncompact manifolds), as well as a good knowledge of symplectic geometry, in connection with the global theory of Fourier integral operators. It is a preference title to have also knowledge of the techniques related to time-frequency analysis and FBI-transform.
Research objectives	We aim at obtaining substantial steps ahead in the construction of a satisfactory global theory of Fourier integral operators on $\mathbf{R}^n$ and other noncompact settings.
Website and Contact	http://www.matematica.unito.it/do/docenti.pl/Show?_id=scorias sandro.coriasco@unito.it, 011-6702803